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Inequalities associated with intra–inter-class correlation matrices

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Abstract

This paper complements the results of Tong (Ann. Statist. 17 (1989) 429), Shaked and Tong (Ann. Statist. 20 (1992) 614) and Eaton (in: Stochastic Inequalities, IMS Lecture Notes Monograph Series, Vol. 22, 1993, 76) by deriving some monotonicity results associated with intra–inter-class correlation matrices. In Section 2, we consider the problem of comparing these matrices in terms of the orderings induced by several subgroups of the orthogonal group. Section 3 is devoted to deriving some probability inequalities for normal random variables whose correlation matrix is of the intra–inter-class correlation structure.

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1. Introduction

Let $Y = (Y_1, \ldots, Y_n)'$ be an $n \times 1$ random vector such that each Y_i has common mean $\beta \in R^1$ and common variance $\sigma^2 > 0$:

$$\mathbf{E}(Y) = \beta 1_n \quad \text{and} \quad \mathbf{Cov}(Y) = \sigma^2 R, \tag{1.1}$$

where $1_n = (1, \ldots, 1)' : n \times 1$ and R is the correlation matrix of Y. This paper treats the case where R belongs to a set of intra–inter-class correlation matrices, and establishes some inequalities which complement the results of Tong [11,13], Shaked

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and Tong [8], Stępniak [10] and Eaton [4]. To state the problem more precisely, let a partition $\mathbf{k}$ of the integer n be an $n \times 1$ vector of nonnegative integers such that

$$\mathbf{k} = (k_1, \dots, k_r, 0, \dots, 0)', \quad k_1 \geqslant \cdots \geqslant k_r \geqslant 1 \quad \text{and} \quad \sum_{i=1}^{r} k_i = n, \tag{1.2}$$

where r is called the length of $\mathbf{k}$. Let $P(n)$ denote the set of all partitions of n of form (1.2). For each $\mathbf{k} \in P(n)$ and for arbitrary but fixed $0 \leqslant \lambda \leqslant \theta < 1$, the intra–inter-class correlation matrix $R(\mathbf{k}) = R(\mathbf{k} : \theta, \lambda)$ is defined by

$$\begin{aligned} R(\mathbf{k}) &= (1-\theta)I_n + (\theta - \lambda)J(\mathbf{k}) + \lambda 1_n 1_n' \quad \text{with} \\ J(\mathbf{k}) &= \operatorname{diag}\{1_{k_1}1_{k_1}', \dots, 1_{k_r}1_{k_r}'\}, \end{aligned} \tag{1.3}$$

where diag denotes the block diagonal matrix. (The term "intra–inter-class correlation" is due to Eaton [4].) Clearly, when we partition $\mathbf{R}(\mathbf{k})$ into r^2 blocks according to $\mathbf{k}$, the (i,i)th block $R_{ii}(\mathbf{k})$ and (i,j)th block $R_{ij}(\mathbf{k})$ $(i \neq j)$ are of the form

$$R_{ii}(\mathbf{k}) = (1-\theta)I_{k_i} + \theta 1_{k_i}1_{k_i}' : k_i \times k_i \quad \text{and} \quad R_{ij}(\mathbf{k}) = \lambda 1_{k_i}1_{k_j}' : k_i \times k_j,$$

respectively. In particular, when $\mathbf{k} = \mathbf{k_{max}}$ or $\mathbf{k_{min}}$, where

$$\mathbf{k_{max}} = (n, 0, \dots, 0)' \in P(n) \quad \text{and} \quad \mathbf{k_{min}} = (1, \dots, 1)' \in P(n), \tag{1.4}$$

the matrices $R(\mathbf{k_{max}})$ and $R(\mathbf{k_{min}})$ take the form

$$R(\mathbf{k_{max}}) = (1-\theta)I_n + \theta 1_n 1_n' \quad \text{and} \quad R(\mathbf{k_{min}}) = (1-\lambda)I_n + \lambda 1_n 1_n',$$

respectively. Note that the two partitions $\mathbf{k_{max}}$ and $\mathbf{k_{min}}$ are the extreme ones in $P(n)$ in terms of the majorization ordering $\succ$, that is,

$$\mathbf{k_{max}} \succ \mathbf{k} \succ \mathbf{k_{min}} \quad \text{for any } \mathbf{k} \in P(n), \tag{1.5}$$

where $\mathbf{k} \succ \mathbf{k}^*$ means that $\mathbf{k}$ majorizes $\mathbf{k}^*$ (see [7] for definition).

Inequality (1.5) raises the question of how $R(\mathbf{k})$'s are ordered by the strength of correlation. Intuitively, $R(\mathbf{k_{max}})$ can be regarded as a "larger" correlation matrix than $R(\mathbf{k_{min}})$ in the sense that $R(\mathbf{k_{max}})$ has more correlation than $R(\mathbf{k_{min}})$, since the off-diagonal elements of $R(\mathbf{k_{max}})$ are uniformly larger than those of $R(\mathbf{k_{min}})$. However, such an elementwise inequality does not generally hold for $R(\mathbf{k})$ and $R(\mathbf{k}^*)$ such that $\mathbf{k} \succ \mathbf{k}^*$. Hence it is not clear whether $\mathbf{k} \succ \mathbf{k}^*$ implies that $R(\mathbf{k})$ has more correlation than $R(\mathbf{k}^*)$ has. The following monotonicity result obtained by Tong [11] gives an answer to this question.

Proposition 1.1 (Tong [11]). *Let $\mathbf{k} \succ \mathbf{k}^*$ and suppose that*

$$\begin{aligned} Y &= (Y_1, \dots, Y_n)' \sim N_n(\beta 1_n, \sigma^2 R(\mathbf{k})) \quad \text{and} \\ Y^* &= (Y_1^*, \dots, Y_n^*)' \sim N_n(\beta 1_n, \sigma^2 R(\mathbf{k}^*)). \end{aligned} \tag{1.6}$$

Then Y is more positively dependent than Y^* in the sense that $\mathbf{E}\{\prod_{i=1}^n f(Y_i)\} \geqslant \mathbf{E}\{\prod_{i=1}^n f(Y_i^*)\}$ holds for all Borel-measurable functions $f : R^1 \to [0, \infty)$ such that the expectations exist.

Although Proposition 1.1 clarifies the relation between $R(\mathbf{k})$ and the strength of positive dependence of Y, it strongly depends on the normality assumption.

On the other hand, by using the notion of group-induced ordering, Eaton [4] introduced an ordering for the covariance matrices of random vectors in the context of comparison of experiments. By applying his result to the set of $R(\mathbf{k})$'s, it can be shown that if $\mathbf{k} \succ \mathbf{k}^*$, then $R(\mathbf{k})$ is smaller than $R(\mathbf{k}^*)$ with respect to the ordering induced by a subgroup of $\mathcal{G}\ell_n$, where $\mathcal{G}\ell_n$ denotes the group of $n \times n$ nonsingular matrices. To see this more exactly, we begin with providing a brief review on the notion of group-induced ordering: Suppose that

$$\mathbf{E}(Y) = \beta 1_n \quad \text{and} \quad \mathbf{Cov}(Y) = \Sigma \quad \text{with } \Sigma \in \mathcal{S}(n), \tag{1.7}$$

where $\mathcal{S}(n)$ denotes the set of $n \times n$ positive definite matrices. For a group $\mathcal{G}$ $(\subset \mathcal{G}\ell_n)$ acting on $\mathcal{S}(n)$ via the group action $\Sigma \to G\Sigma G'$, let $C[\Sigma : \mathcal{G}]$ be the convex hull of the $\mathcal{G}$-orbit of Σ, that is, the convex hull of the set $\{G\Sigma G' \mid G \in \mathcal{G}\}$. For two matrices $\Sigma, \Sigma^* \in \mathcal{S}(n)$, we write

$$\Sigma \leqslant_{\mathcal{G}} \Sigma^* \quad \text{if } \Sigma \in C[\Sigma^* : \mathcal{G}].$$

The pre-ordering $\leqslant_{\mathcal{G}}$ thus defined is called the group-induced ordering, or the ordering induced by $\mathcal{G}$. The result of Eaton [4] (modified to the setup (1.7)) is the following: Let $\mathcal{G}\ell_n(1_n)$ be the group of $n \times n$ nonsingular matrices G satisfying $G1_n = 1_n$, that is,

$$\mathcal{G}\ell_n(1_n) = \{G \in \mathcal{G}\ell_n \mid G1_n = 1_n\}.$$

Then $\mathcal{G}\ell_n(1_n)$ acts on the spaces of Y and Σ via $Y \to GY$ and $\Sigma \to G\Sigma G'$, respectively. He showed that for $\Sigma, \Sigma^* \in \mathcal{S}(n)$, the inequality $\phi(\Sigma) \geqslant \phi(\Sigma^*)$ with $\phi(\Sigma) = (1_n{}'\Sigma^{-1}1_n)^{-1}$ is equivalent to $\Sigma \leqslant_{\mathcal{G}\ell_n(1_n)} \Sigma^*$. Here the above inequality for ϕ has been fully investigated by many authors in the context of comparison of experiments. Among others, Shaked and Tong [8], Eaton [4], Stępniak [10] and Tong [13] showed from various points of view that $\mathbf{k} \succ \mathbf{k}^*$ implies $\phi(R(\mathbf{k})) \geqslant \phi(R(\mathbf{k}^*))$. Thus under condition (1.7), it is obtained that

$$R(\mathbf{k}) \leqslant_{\mathcal{G}\ell_n(1_n)} R(\mathbf{k}^*) \quad \text{whenever } \mathbf{k} \succ \mathbf{k}^*, \tag{1.8}$$

where $R(\mathbf{k})$'s are viewed as the elements of $\mathcal{S}(n)$.

In Section 2, we complement result (1.8) by dealing with the ordering induced by some appropriate subgroups of the group $\mathcal{O}_n$ of $n \times n$ orthogonal matrices. While the ordering $\leqslant_{\mathcal{G}\ell_n(1_n)}$ is natural for positive definite matrices, it is not necessarily suitable for comparison of correlation matrices. In fact, for a correlation matrix R, the action of $\mathcal{G}\ell_n(1_n)$ does not necessarily satisfy $tr(R) = tr(GRG')$. Furthermore, the identity matrix I_n, which is an extreme correlation matrix, is not necessarily an extreme element with respect to the ordering $\leqslant_{\mathcal{G}\ell_n(1_n)}$. On the contrary, $tr(R) = tr(\Gamma R \Gamma')$ is clearly satisfied for any $\Gamma \in \mathcal{O}_n$, and, as will be seen in the next section, the matrix I_n is minimal with respect to the ordering induced by any subgroup of $\mathcal{O}_n$. Section 3 is devoted to deriving several probability inequalities for the quantities $|Y_i - \beta|$'s which complement the results of Section 2 and Proposition 1.1.

2. Orderings induced by subgroups of $\mathcal{O}_n$

Let $Y = (Y_1, \dots, Y_n)'$ be an $n \times 1$ random vector such that

$$\mathbf{E}(Y) = \beta 1_n \quad \text{and} \quad \mathbf{Cov}(Y) = \sigma^2 \Sigma \quad \text{with } \Sigma \in \mathcal{T}(n), \tag{2.1}$$

where $\mathcal{T}(n)$ is a subset of $\mathcal{S}(n)$ defined as

$$\mathcal{T}(n) = \{\Sigma \in \mathcal{S}(n) \mid tr(\Sigma) = n\}.$$

Then clearly the intra–inter-correlation matrices $R(\mathbf{k})$'s are contained in $\mathcal{T}(n)$. This section is concerned with comparing the matrices $R(\mathbf{k})$'s in terms of the ordering induced by several subgroups of $\mathcal{O}_n$. Note here that if two groups $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfying $\mathcal{G}_1 \subset \mathcal{G}_2$ act on $\mathcal{T}(n)$ and if $\Sigma \leqslant_{\mathcal{G}_1} \Sigma^*$, then $\Sigma \leqslant_{\mathcal{G}_2} \Sigma^*$ holds since $C[\Sigma^* : \mathcal{G}_1] \subset C[\Sigma^* : \mathcal{G}_2]$. This fact will be often used in this section.

For any $n \times r$ matrix X such that $X'X = I_r$, let $\mathcal{O}_n(X)$ be the group of $n \times n$ orthogonal matrices Γ satisfying $\Gamma X = X$, that is,

$$\mathcal{O}_n(X) = \{\Gamma \in \mathcal{O}_n \mid \Gamma X = X\}. \tag{2.2}$$

Note here that the assumption $X'X = I_r$ does not lose any generality, since the condition $\Gamma X = X$ is equivalent to $\Gamma X G = XG$ for any $G \in \mathcal{G}\ell_n$. In other words, $\mathcal{O}_n(X)$ depends on X only through the linear subspace $L(X)$ spanned by the column vectors of X. Therefore, when $X = e$ with

$$e = 1_n/\sqrt{n},$$

the group $\mathcal{O}_n(e)$ is the group of orthogonal matrices Γ satisfying $\Gamma 1_n = 1_n$.

The first problem considered in this section is to find an expression of $R(\mathbf{k})$ which clarifies that $R(\mathbf{k})$ is a minimal element with respect to the ordering $\leqslant_{\mathcal{O}_n(X)}$ induced by $\mathcal{O}_n(X)$ for appropriately chosen X. More specifically, for each $\mathbf{k} = (k_1, \dots, k_r, 0, \dots, 0)' \in P(n)$, we show that there exists an $n \times r$ matrix $X = X(\mathbf{k})$ such that

(C1) any $\Gamma \in \mathcal{O}_n(X)$ satisfies $\Gamma 1_n = 1_n$, (which guarantees that $\mathcal{O}_n(X)$ acts on the space $\mathcal{T}(n)$ via $\Sigma \to \Gamma \Sigma \Gamma'$),
(C2) $R(\mathbf{k})$ is a minimal element of $\mathcal{T}(n)$ with respect to the ordering $\leqslant_{\mathcal{O}_n(X)}$.

Here, $R(\mathbf{k})$ is said to be minimal with respect to the ordering $\leqslant_{\mathcal{O}_n(X)}$, if no matrix $R \in \mathcal{T}(n)$ satisfies

$$R \leqslant_{\mathcal{O}_n(X)} R(\mathbf{k}). \tag{2.3}$$

In order to show that $R(\mathbf{k})$ is minimal, it is sufficient to prove that $R(\mathbf{k})$ satisfies

$$\Gamma R(\mathbf{k}) \Gamma' = R(\mathbf{k}) \quad \text{for any } \Gamma \in \mathcal{O}_n(X), \tag{2.4}$$

since the above equality implies that $C[R(\mathbf{k}) : \mathcal{O}_n(X)] = \{R(\mathbf{k})\}$. To this end, we need the following two lemmas.

Lemma 2.1. *For given $X : n \times r$ such that $X'X = I_r$, let Z be an $n \times (n - r)$ matrix satisfying*

$$X'Z = 0 \quad \textit{and} \quad Z'Z = I_{n-r}.$$

Then the group $\mathcal{O}_n(X)$ is described as

$$\mathcal{O}_n(X) = \left\{ \Gamma \equiv (X, Z) \begin{pmatrix} I_r & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} X' \\ Z' \end{pmatrix} \,\middle|\, \Psi \in \mathcal{O}_{n-r} \right\}. \tag{2.5}$$

Proof. It is clear that the right-hand side of (2.5) is contained in $\mathcal{O}_n(X)$. We show the converse. Since the matrix $W \equiv (X, Z)$ is in $\mathcal{O}_n$, any $\Gamma \in \mathcal{O}(X)$ can be rewritten as

$$\Gamma = WW'\Gamma WW' = W\tilde{\Gamma}W' = (X, Z) \begin{pmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} \\ \tilde{\Gamma}_{21} & \tilde{\Gamma}_{22} \end{pmatrix} \begin{pmatrix} X' \\ Z' \end{pmatrix}$$

with $\tilde{\Gamma} \equiv W'\Gamma W \equiv (\tilde{\Gamma}_{ij})$, where the matrices $\tilde{\Gamma}_{ij}$'s are determined uniquely for given X and Z. Here the condition $\Gamma X = X$ implies

$$\tilde{\Gamma}_{11} = I_k \quad \text{and} \quad \tilde{\Gamma}_{21} = 0.$$

Furthermore, we have $\tilde{\Gamma}_{12} = 0$ and $\tilde{\Gamma}_{22} \in \mathcal{O}_{n-r}$, since $\tilde{\Gamma}$ is in $\mathcal{O}_n$. $\square$

Lemma 2.2. *Let $\Sigma \in \mathcal{S}(n)$. For given $X : n \times r$ such that $X'X = I_r$, the matrix Σ satisfies*

$$\Gamma\Sigma\Gamma' = \Sigma \quad \textit{for any } \Gamma \in \mathcal{O}_n(X), \tag{2.6}$$

if and only if Σ is of the form

$$\Sigma = (X, Z) \begin{pmatrix} \Upsilon & 0 \\ 0 & \gamma I_{n-r} \end{pmatrix} \begin{pmatrix} X' \\ Z' \end{pmatrix} \quad \textit{for some } \Upsilon \in \mathcal{S}(r) \textit{ and } \gamma > 0,$$

where the definition of the matrix Z is same as that in Lemma 2.1.

Proof. As is done in the proof of Lemma 2.1, let $W = (X, Z)$ and rewrite $\Sigma \in \mathcal{S}(n)$ as

$$\Sigma = WW'\Sigma WW' = (X, Z) \begin{pmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{pmatrix} \begin{pmatrix} X' \\ Z' \end{pmatrix} \quad \text{(say)}.$$

Since any $\Gamma \in \mathcal{O}_n(X)$ is written as

$$\Gamma = (X, Z) \begin{pmatrix} I_r & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} X' \\ Z' \end{pmatrix} \quad \text{with } \Psi \in \mathcal{O}_{n-r},$$

condition (2.6) is equivalent to

$$\Psi\tilde{\Sigma}_{21} = \tilde{\Sigma}_{21} \quad \text{and} \quad \Psi\tilde{\Sigma}_{22}\Psi' = \tilde{\Sigma}_{22} \quad \text{for any } \Psi \in \mathcal{O}_{n-r},$$

which is in turn equivalent to

$$\tilde{\Sigma}_{21} = 0 = \tilde{\Sigma}'_{12} \quad \text{and} \quad \tilde{\Sigma}_{22} = \gamma I_{n-r} \quad \text{for some } \gamma > 0,$$

since $\tilde{\Sigma} \in \mathcal{S}(n)$. This completes the proof. $\square$

Now for each $\mathbf{k} = (k_1, \ldots, k_r, 0, \ldots, 0)' \in P(n)$, let

$$X = X(\mathbf{k}) = \mathrm{diag}\{1_{k_1}/\sqrt{k_1}, \ldots, 1_{k_r}/\sqrt{k_r}\} : n \times r \tag{2.7}$$

and let $Z = Z(\mathbf{k})$ be any $n \times (n - r)$ matrix such that

$$X'Z = 0 \quad \text{and} \quad Z'Z = I_{n-r}.$$

Then the matrix $(X, Z) = (X(\mathbf{k}), Z(\mathbf{k}))$ is an $n \times n$ orthogonal matrix. Note here that any $\Gamma \in \mathcal{O}_n(X)$ satisfies $\Gamma 1_n = 1_n$, since

$$1_n = Xs \quad \text{with } s = s(\mathbf{k}) = (\sqrt{k_1}, \ldots, \sqrt{k_r})' : r \times 1. \tag{2.8}$$

This shows that $\mathcal{O}_n(X)$ satisfies condition (C1). Condition (C2) is proved by the following theorem.

Theorem 2.3. *For any $\mathbf{k} \in P(n)$, the matrix $R(\mathbf{k})$ is expressed as*

$$R(\mathbf{k}) = (X, Z)\begin{pmatrix} U & 0 \\ 0 & (1-\theta)I_{n-r} \end{pmatrix}\begin{pmatrix} X' \\ Z' \end{pmatrix} \tag{2.9}$$

with $X = X(\mathbf{k})$, $Z = Z(\mathbf{k})$ and

$$U = U(\mathbf{k}) = (1-\theta)I_r + (\theta - \lambda)N + \lambda ss',$$

where $s = s(\mathbf{k})$ is given in (2.8) *and*

$$N = N(\mathbf{k}) = \mathrm{diag}\{k_1, \ldots, k_r\} : r \times r.$$

Hence $R(\mathbf{k})$ is a minimal element with respect to the ordering induced by $O_n(X(\mathbf{k}))$.

Proof. Rewriting the right-hand side of $R(\mathbf{k}) = (1-\theta)I_n + (\theta-\lambda)J(\mathbf{k}) + \lambda 1_n 1_n'$ by using the following three equalities

$$I_n = XX' + ZZ', \quad J(\mathbf{k}) = XNX' \quad \text{and} \quad 1_n = Xs$$

yields

$$\begin{aligned} R(\mathbf{k}) &= (1-\theta)[XX' + ZZ'] + (\theta-\lambda)XNX' + \lambda Xss'X' \\ &= X[(1-\theta)I_r + (\theta-\lambda)N + \lambda ss']X' + Z[(1-\theta)I_{n-r}]Z', \end{aligned}$$

from which (2.9) follows. By Lemma 2.2, (2.9) is equivalent to (2.4). Hence we see that $R(\mathbf{k})$ is minimal with respect to the ordering induced by $\mathcal{O}_n(X)$. $\square$

Two extreme cases are examined here. When $\mathbf{k} = \mathbf{k_{max}}$, then $X(\mathbf{k_{max}}) = e$ and the matrix $R(\mathbf{k_{max}}) = (1-\theta)I_n + \theta 1_n 1_n'$ is minimal with respect to the ordering induced by $\mathcal{O}_n(e)$. The expression (2.9) for $R(\mathbf{k_{max}})$ is

$$R(\mathbf{k_{max}}) = (e, Z)\begin{pmatrix} U & 0 \\ 0 & (1-\theta)I_{n-1} \end{pmatrix}\begin{pmatrix} e' \\ Z' \end{pmatrix}$$

with

$$U = U(\mathbf{k_{max}}) = (1-\theta) + (\theta-\lambda)n + \lambda n = (1-\theta) + \theta n.$$

On the other hand, when $\mathbf{k} = \mathbf{k}_{\mathbf{min}}$, we have $X(\mathbf{k}_{\mathbf{min}}) = I_n$, $N(\mathbf{k}_{\mathbf{min}}) = I_n$, $s(\mathbf{k}_{\mathbf{min}}) = 1_n$ and $\mathcal{O}_n(X(\mathbf{k}_{\mathbf{min}})) = \{I_n\}$. While the two matrices $R(\mathbf{k}_{\mathbf{max}})$ and $R(\mathbf{k}_{\mathbf{min}})$ have the same structure, Theorem 2.3 distinguishes them by the two different matrices $X(\mathbf{k}_{\mathbf{max}})$ and $X(\mathbf{k}_{\mathbf{min}})$.

It follows from (2.8) that for any $\mathbf{k} \in P(n)$,

$$L(e) = L(X(\mathbf{k}_{\mathbf{max}})) \subset L(X(\mathbf{k})) \subset L(X(\mathbf{k}_{\mathbf{min}})) = R^n,$$

which in turn implies that for any $\mathbf{k} \in P(n)$,

$$\mathcal{O}_n(e) = \mathcal{O}_n(X(\mathbf{k}_{\mathbf{max}})) \supset \mathcal{O}_n(X(\mathbf{k})) \supset \mathcal{O}_n(X(\mathbf{k}_{\mathbf{min}})) = \{I_n\}. \tag{2.10}$$

By combining (2.10) with Theorem 2.3, we can classify $R(\mathbf{k})$'s according to the minimality with respect to the orderings induced by $\mathcal{O}_n(X(\mathbf{k}))$'s.

Relation (2.10) suggests the problem of comparing $R(\mathbf{k})$'s in terms of the ordering induced by $\mathcal{O}_n(e)$. To this end, for each $\mathbf{k} \in P(n)$, we derive the smallest element, say $\tilde{R}(\mathbf{k})$, of the convex set $C[R(\mathbf{k}) : \mathcal{O}_n(e)]$ in the sense that $\tilde{R}(\mathbf{k})$ satisfies

$$\tilde{R}(\mathbf{k}) \leqslant_{\mathcal{O}_n(e)} R \quad \text{for any } R \in C[R(\mathbf{k}) : \mathcal{O}_n(e)].$$

Since $\mathcal{O}_n(e)$ is a compact group, there exists the unique $\mathcal{O}_n(e)$-invariant probability measure v (the uniform distribution on $\mathcal{O}_n(e)$), and the matrix $\tilde{R}(\mathbf{k})$ is given by

$$\tilde{R}(\mathbf{k}) = \int_{\mathcal{O}_n(e)} \Gamma R(\mathbf{k}) \Gamma' v(d\Gamma).$$

(See Eaton ([3, p. 5]) for general theory.)

Theorem 2.4. *For each* $\mathbf{k} \in P(n)$, *the smallest element* $\tilde{R}(\mathbf{k})$ *of* $C[R(\mathbf{k}) : \mathcal{O}_n(e)]$ *is given by*

$$\tilde{R}(\mathbf{k}) = (1 - \tilde{\theta}) I_n + \tilde{\theta} 1_n 1_n', \tag{2.11}$$

where $\tilde{\theta} = \tilde{\theta}(\mathbf{k})$ *is defined as*

$$\begin{aligned} \tilde{\theta}(\mathbf{k}) &= \frac{1}{n(n-1)} \{1_n' R(\mathbf{k}) 1_n - n\} \\ &= a(\mathbf{k})\theta + (1 - a(\mathbf{k}))\lambda \end{aligned} \tag{2.12}$$

with

$$a(\mathbf{k}) = \frac{1}{n-1} \left\{ \frac{1}{n} \sum_{i=1}^{r} k_i^2 - 1 \right\}. \tag{2.13}$$

The functions $a(\mathbf{k})$ *and* $\tilde{\theta}(\mathbf{k})$ *are strictly Schur-convex, and hence* $a(\mathbf{k}) \in [0, 1]$ *and* $\tilde{\theta}(\mathbf{k}) \in [\lambda, \theta]$.

Here $a(\mathbf{k})$ defined on $P(n)$ is called Schur-convex if $\mathbf{k} \succ \mathbf{k}^*$ implies $a(\mathbf{k}) \geqslant a(\mathbf{k}^*)$. If $a(\mathbf{k}) > a(\mathbf{k}^*)$ whenever $\mathbf{k} \succ \mathbf{k}^*$ and $\mathbf{k} \neq \mathbf{k}^*$, then $a(\mathbf{k})$ is said to be strictly Schur-convex. (See [7, p. 54]).

Proof. Since, by Lemma 2.1, any $\Gamma \in \mathcal{O}_n(e)$ is of the form

$$\Gamma = (e, Z)\begin{pmatrix} 1 & 0 \\ 0 & \Psi \end{pmatrix}\begin{pmatrix} e' \\ Z' \end{pmatrix} \quad \text{with } \Psi \in \mathcal{O}_{n-1},$$

where $Z : n \times (n-1)$ satisfies $e'Z = 0$ and $Z'Z = I_{n-1}$, the matrix $\Gamma R \Gamma'$ with $R = R(\mathbf{k})$ is expressed as

$$\begin{aligned}\Gamma R \Gamma' &= (e, Z)\begin{pmatrix} e'Re & e'RZ\Psi' \\ \Psi Z'Re & \Psi Z'RZ\Psi' \end{pmatrix}\begin{pmatrix} e' \\ Z' \end{pmatrix} \\ &= (e'Re)ee' + ee'RZ\Psi'Z' + Z\Psi Z'Ree' + Z\Psi Z'RZ\Psi'Z'.\end{aligned}$$

That Γ is distributed as the uniform distribution ν is equivalent to that Ψ is distributed as the uniform distribution on $\mathcal{O}_{n-1}$. Therefore,

$$\begin{aligned}\int_{\mathcal{O}_n(e)} \Gamma R \Gamma' \nu(d\Gamma) &= (e'Re)ee' + ee'RZ\mathbf{E}\{\Psi\}'Z' + Z\mathbf{E}\{\Psi\}Z'Ree' \\ &\quad + Z\mathbf{E}\{\Psi Z'RZ\Psi'\}Z' \\ &= (e'Re)ee' + \frac{tr(Z'RZ)}{n-1}ZZ' \\ &= \frac{tr(Z'RZ)}{n-1}I_n + \frac{1}{n}\left\{\frac{1}{n}1_n{}'R1_n - \frac{tr(Z'RZ)}{n-1}\right\}1_n1_n{}',\end{aligned}$$

where $\mathbf{E}$ in the first line denotes the expectation with respect to the distribution of Ψ, the second line follows from the following two equalities:

$$\mathbf{E}\{\Psi\} = 0 \quad \text{and} \quad \mathbf{E}\{\Psi A \Psi'\} = \frac{tr(A)}{n-1}I_{n-1} \quad \text{for any } A : (n-1) \times (n-1),$$

and the third from the matrix identity $ee' + ZZ' = (1/n)1_n1_n{}' + ZZ' = I_n$. Noting that

$$tr(Z'RZ) = tr\left(R\left[I_n - \frac{1}{n}1_n1_n{}'\right]\right) = tr(R) - \frac{1}{n}1_n{}'R1_n = n - \frac{1}{n}1_n{}'R1_n,$$

$$1_n{}'R1_n = (1-\theta)n + (\theta - \lambda)\sum_{i=1}^{r} k_i^2 + \lambda n^2 \quad \text{(obtained directly from (1.3))}$$

yields results (2.11)–(2.13). The strict Schur-convexity of $\sum_{i=1}^{r} k_i^2$ is well-known (see, for example, [7, p. 64]), which implies that both $a(\mathbf{k})$ and $\tilde{\theta}(\mathbf{k})$ are also strictly Schur-convex. $\square$

Thus for each $R(\mathbf{k})$, we obtain the correlation matrix $\tilde{R}(\mathbf{k}) \in \mathcal{T}(n)$ such that $\tilde{R}(\mathbf{k}) \leqslant_{\mathcal{O}_n(e)} R(\mathbf{k})$. Here $\tilde{R}(\mathbf{k})$ is minimal with respect to the ordering $\leqslant_{\mathcal{O}_n(e)}$.

Corollary 2.5. *Let* $\mathbf{k} \succ \mathbf{k}^*$ *and* $\mathbf{k} \neq \mathbf{k}^*$. *Then neither* $R(\mathbf{k}) \leqslant_{\mathcal{O}_n(e)} R(\mathbf{k}^*)$ *nor* $R(\mathbf{k}) \geqslant_{\mathcal{O}_n(e)} R(\mathbf{k}^*)$ *holds.*

Proof. Suppose first that $R(\mathbf{k}) \leqslant_{\mathcal{O}_n(e)} R(\mathbf{k}^*)$. Then $R(\mathbf{k})$ is expressed as $R(\mathbf{k}) = \sum_{i=1}^m c_i \Gamma_i R(\mathbf{k}^*) \Gamma_i'$ for some integer m and $\Gamma_i \in \mathcal{O}_n(e)$ $(i = 1, \dots, m)$, since $R(\mathbf{k}) \in C[R(\mathbf{k}^*) : \mathcal{O}_n(e)]$. This implies that

$$\begin{aligned} \tilde{R}(\mathbf{k}) &= \int_{\mathcal{O}_n(e)} \Gamma R(\mathbf{k}) \Gamma' v(d\Gamma) = \sum_{i=1}^m c_i \int_{\mathcal{O}_n(e)} \Gamma \Gamma_i R(\mathbf{k}^*) \Gamma_i' \Gamma' v(d\Gamma) \\ &= \sum_{i=1}^m c_i \int_{\mathcal{O}_n(e)} \Gamma R(\mathbf{k}^*) \Gamma' v(d\Gamma) = \tilde{R}(\mathbf{k}^*), \end{aligned}$$

where the third equality follows since v is invariant under $\mathcal{O}_n(e)$. Thus we obtain $\tilde{\theta}(\mathbf{k}) = \tilde{\theta}(\mathbf{k}^*)$, which contradicts the strict Schur-convexity of $\tilde{\theta}(\mathbf{k})$. By interchanging the role of $R(\mathbf{k})$ and $R(\mathbf{k}^*)$, it is also shown that $R(\mathbf{k}) \geqslant_{\mathcal{O}_n(e)} R(\mathbf{k}^*)$ does not hold. □

This fact leads to the problem of comparing the matrices in terms of the ordering $\leqslant_{\mathcal{O}_n}$ induced by $\mathcal{O}_n$ which is larger than $\mathcal{O}_n(e)$. As is proved in Eaton [2], for two matrices $\Sigma, \Sigma^* \in \mathcal{T}(n)$, the condition $\Sigma \geqslant_{\mathcal{O}_n} \Sigma^*$ is equivalent to

$$\rho(\Sigma) \succ \rho(\Sigma^*),$$

where $\rho(\Sigma) = (\rho_1(\Sigma), \dots, \rho_n(\Sigma))' : n \times 1$ is the vector of the latent roots of Σ such that $\rho_1(\Sigma) \geqslant \cdots \geqslant \rho_n(\Sigma)$. (Although $\Gamma \in \mathcal{O}_n$ does not necessarily satisfy $\Gamma \beta 1_n = \beta 1_n$ in (2.1) unless $\beta = 0$, the ordering $\leqslant_{\mathcal{O}_n}$ for $\Sigma \in \mathcal{T}(n)$, or equivalently, the majorization ordering for $\rho(\Sigma)$ is of independent interest.)

Proposition 2.6. *If* $\mathbf{k} \succ \mathbf{k}^*$, *then* $\rho(\tilde{R}(\mathbf{k})) \succ \rho(\tilde{R}(\mathbf{k}^*))$.

Proof. In general, the latent roots of the correlation matrix $C_\theta = (1-\theta) I_n + \theta 1_n 1_n'$ such that $\theta \geqslant 0$ is given by

$$\rho(C_\theta) = ((1-\theta) + n\theta, 1-\theta, \dots, 1-\theta)', \tag{2.14}$$

and it is easy to see that $\theta \geqslant \theta^*$ is equivalent to $\rho(C_\theta) \succ \rho(C_{\theta^*})$. By applying this fact to $\tilde{R}(\mathbf{k})$'s and using the Schur-convexity of $\tilde{\theta}(\mathbf{k})$, the result follows. □

Thus for any $0 \leqslant \lambda \leqslant \theta < 1$ and for any $\mathbf{k}, \mathbf{k}^* \in P(n)$ such that $\mathbf{k} \succ \mathbf{k}^*$, we have

$$\begin{aligned} \rho(R(\mathbf{k_{max}})) &= \rho(\tilde{R}(\mathbf{k_{max}})) \succ \rho(\tilde{R}(\mathbf{k})) \succ \rho(\tilde{R}(\mathbf{k}^*)) \succ \rho(\tilde{R}(\mathbf{k_{min}})) \\ &= \rho(R(\mathbf{k_{min}})). \end{aligned} \tag{2.15}$$

Next we consider the ordering for $R(\mathbf{k})$'s. For $R(\mathbf{k_{max}})$ and $R(\mathbf{k})$, we have

Proposition 2.7. $\rho(R(\mathbf{k_{max}})) \succ \rho(R(\mathbf{k}))$ *holds for any* $\mathbf{k} \in P(n)$.

Proof. An equivalent statement $R(\mathbf{k_{max}}) \geqslant_{\mathcal{O}_n} R(\mathbf{k})$ is proved here. Let $e_{(i)} : n \times 1$ be the ith column vector of $X = X(\mathbf{k})$. Then $R(\mathbf{k_{max}})$ and $R(\mathbf{k})$ are expressed as

$$R(\mathbf{k_{max}}) = (1-\theta) I_n + \theta n e e',$$

$$
\begin{aligned}
R(\mathbf{k}) &= (1-\theta)I_n + (\theta-\lambda)XNX' + \lambda nee' \quad \text{with } N = N(\mathbf{k}) \\
&= (1-\theta)I_n + (\theta-\lambda)\sum_{i=1}^{r} k_i e_{(i)}e_{(i)}{}' + \lambda nee',
\end{aligned} \tag{2.16}
$$

respectively. Here, let $\Gamma_i \in \mathcal{O}_n$ $(i = 1, \ldots, r+1)$ be orthogonal matrices such that

$$\Gamma_1 e = e_{(1)}, \ldots, \Gamma_r e = e_{(r)} \quad \text{and} \quad \Gamma_{r+1} e = e,$$

and let

$$c_1 = \frac{k_1(\theta-\lambda)}{n\theta}, \ldots, c_r = \frac{k_r(\theta-\lambda)}{n\theta} \quad \text{and} \quad c_{r+1} = \frac{\lambda}{\theta}.$$

Then $0 \leqslant c_i \leqslant 1$ $(i = 1, \ldots, r+1)$, $\sum_{i=1}^{r+1} c_i = 1$ and

$$
\begin{aligned}
\sum_{i=1}^{r+1} c_i \Gamma_i R(\mathbf{k_{max}}) \Gamma_i{}' &= (1-\theta)I_n + \theta n \sum_{i=1}^{r+1} c_i \Gamma_i ee' \Gamma'_i \\
&= (1-\theta)I_n + \theta n \left\{ \sum_{i=1}^{r} c_i e_{(i)} e_{(i)}{}' + c_{r+1} ee' \right\} \\
&= \text{the right-hand side of (2.16)},
\end{aligned}
$$

which shows that $R(\mathbf{k}) \in C[R(\mathbf{k_{max}}) : \mathcal{O}_n]$. $\square$

In this paper, the question of whether $\mathbf{k} \succ \mathbf{k}^*$ implies $\rho(R(\mathbf{k})) \succ \rho(R(\mathbf{k}^*))$ is left unsolved. However, in the following theorem, upper and lower bounds for $R(\mathbf{k})$ are obtained:

Theorem 2.8. *For any $\mathbf{k} \in P(n)$, the vector $\rho(R(\mathbf{k}))$ is bounded from above and below by*

$$u(\mathbf{k}) \succ \rho(R(\mathbf{k})) \succ l(\mathbf{k}), \tag{2.17}$$

where $u(\mathbf{k})$ and $l(\mathbf{k})$ are defined by

$$
\begin{aligned}
u(\mathbf{k}) &= (1-\theta)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + (\theta-\lambda)\begin{pmatrix} k_1 \\ \vdots \\ k_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&= (1-\theta)\mathbf{k_{min}} + (\theta-\lambda)\mathbf{k} + \lambda \mathbf{k_{max}},
\end{aligned} \tag{2.18}
$$

$$l(\mathbf{k}) = (1-\tilde{\theta}(\mathbf{k}))\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \tilde{\theta}(\mathbf{k})\begin{pmatrix} n \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (1-\tilde{\theta}(\mathbf{k}))\mathbf{k_{min}} + \tilde{\theta}(\mathbf{k})\mathbf{k_{max}}, \tag{2.19}$$

respectively. Furthermore, both $u(\mathbf{k})$ *and* $l(\mathbf{k})$ *are monotone in the majorization ordering in the sense that* $\mathbf{k} \succ \mathbf{k}^*$ *implies* $u(\mathbf{k}) \succ u(\mathbf{k}^*)$ *and* $l(\mathbf{k}) \succ l(\mathbf{k}^*)$.

Proof. Applying Theorem G.1.c. of Marshall and Olkin ([7, p. 243]) to $R(\mathbf{k}) = (1-\theta)I_n + (\theta - \lambda)J(\mathbf{k}) + \lambda 1_n 1_n'$ yields

$$\rho(R(\mathbf{k})) \prec \rho((1-\theta)I_n + (\theta-\lambda)J(\mathbf{k})) + \rho(\lambda 1_n 1_n'),$$

where the right-hand side is equal to $u(\mathbf{k})$. The lower bound $l(\mathbf{k})$ is obtained from Theorem 2.4. That is, Theorem 2.4 implies that $R(\mathbf{k}) \geqslant_{\mathcal{O}_n} \tilde{R}(\mathbf{k})$, or equivalently $\rho(R(\mathbf{k})) \succ \rho(\tilde{R}(\mathbf{k}))$, the right-hand side of which is equal to $l(\mathbf{k})$. The monotonicity of $u(\mathbf{k})$ and $l(\mathbf{k})$ is clear. $\square$

Corollary 2.9. $\rho(R(\mathbf{k})) \succ \rho(R(\mathbf{k}_{\mathbf{min}}))$ *holds for any* $\mathbf{k} \in P(n)$.

Proof. Combining (2.15) with Theorem 2.8 yields

$$\rho(R(\mathbf{k})) \succ \rho(\tilde{R}(\mathbf{k})) \succ \rho(\tilde{R}(\mathbf{k}_{\mathbf{min}})) = \rho(R(\mathbf{k}_{\mathbf{min}})). \qquad \square$$

Finally, two complementary facts related to the results of Stępniak [10] and Giovagnolli and Romanazzi [5] are given. In [10], it is proved that $\phi(R(\mathbf{k})) \geqslant \phi(R(\mathbf{k}^*))$ with $R(\mathbf{k}) = R(\mathbf{k} : \theta, \lambda)$ is equivalent to the one with $\lambda = 0$, that is,

$$\phi(R(\mathbf{k} : \theta, \lambda)) \geqslant \phi(R(\mathbf{k}^* : \theta, \lambda)) \quad \text{if and only if } \phi(R(\mathbf{k} : \theta, 0)) \geqslant \phi(R(\mathbf{k}^* : \theta, 0)),$$

where $R(\mathbf{k} : \theta, 0) = (1-\theta)I_n + \theta J(\mathbf{k})$. Since

$$\rho(R(\mathbf{k} : \theta, 0)) = ((1-\theta) + \theta k_1, \ldots, (1-\theta) + \theta k_r, 1-\theta, \ldots, 1-\theta),$$

we obtain

$$\rho(R(\mathbf{k} : \theta, 0)) \succ \rho(R(\mathbf{k}^* : \theta, 0)) \quad \text{whenever } \mathbf{k} \succ \mathbf{k}^*.$$

On the other hand, Giovagnolli and Romanazzi [5] defined a group-induced ordering for correlation matrices based on the group $\mathcal{G}_n$ generated by $\mathcal{D}_n \cup \mathcal{P}_n$, where $\mathcal{D}_n$ denotes the group of $n \times n$ sign-change matrices and $\mathcal{P}_n$ the group of $n \times n$ permutation matrices. The group $\mathcal{G}_n$ is clearly a subgroup of $\mathcal{O}_n$. In Proposition 1(a) of their paper, a necessary and sufficient condition for a correlation matrix R to satisfy $(1-\theta)I_n + \theta 1_n 1_n' \leqslant_{\mathcal{G}} R$ is obtained as

$$\frac{1}{n(n-1)} \min_{u \in \mathcal{U}} \{u'Ru - n\} \leqslant \theta \leqslant \frac{1}{n(n-1)} \max_{u \in \mathcal{U}} \{u'Ru - n\},$$

where $\mathcal{U} = \{u \in R^n \mid u = (\pm 1, \ldots, \pm 1)'\}$. Since all the elements of $R(\mathbf{k})$ are nonnegative, we have

$$\frac{1}{n(n-1)} \max_{u \in \mathcal{U}} \{u'R(\mathbf{k})u - n\} = \frac{1}{n(n-1)} \{1_n' R(\mathbf{k}) 1_n - n\} = \tilde{\theta}(\mathbf{k}),$$

which in turn implies that

$$\tilde{R}(\mathbf{k}) = (1-\tilde{\theta}(\mathbf{k}))I_n + \tilde{\theta}(\mathbf{k})1_n1_n{}' \leqslant_{\mathscr{G}} R(\mathbf{k}),$$

where $\tilde{R}(\mathbf{k})$ is the correlation matrix obtained in Theorem 2.4.

3. Probability inequalities derived under normality assumption

This section complements Proposition 1.1 by deriving some monotonicity results on the quantities $|Y_i - \beta|$'s under normality assumption. Suppose that

$$Y = (Y_1, \ldots, Y_n)' \sim N_n(\beta 1_n, \sigma^2 R(\mathbf{k})) \quad \text{with } R(\mathbf{k}) = R(\mathbf{k} : \theta, \lambda). \tag{3.1}$$

Tong [13] stated that how to find the inverse matrix $R(\mathbf{k})^{-1}$ is not yet known. However it is easily obtained from Theorem 2.3 as

Lemma 3.1. *For any $\mathbf{k} \in P(n)$, the matrix $R(\mathbf{k})^{-1}$ is expressed as*

$$R(\mathbf{k})^{-1} = (X, Z)\begin{pmatrix} U^{-1} & 0 \\ 0 & \frac{1}{1-\theta} I_{n-r} \end{pmatrix}\begin{pmatrix} X' \\ Z' \end{pmatrix} \tag{3.2}$$

with $X = X(\mathbf{k})$, $Z = Z(\mathbf{k})$ and

$$U^{-1} = U(\mathbf{k})^{-1} = \frac{1}{1-\theta} I_r - \frac{\theta-\lambda}{1-\theta}\tilde{N} - \frac{\lambda}{1+\lambda G(\mathbf{k})}\tilde{s}\tilde{s}',$$

where

$$\tilde{N} = \tilde{N}(\mathbf{k}) = \mathrm{diag}\left\{\frac{k_1}{(1-\theta)+(\theta-\lambda)k_1}, \ldots, \frac{k_r}{(1-\theta)+(\theta-\lambda)k_r}\right\} : r \times r,$$

$$\tilde{s} = \tilde{s}(\mathbf{k}) = \left(\frac{\sqrt{k_1}}{(1-\theta)+(\theta-\lambda)k_1}, \ldots, \frac{\sqrt{k_r}}{(1-\theta)+(\theta-\lambda)k_r}\right) : r \times 1,$$

$$G(\mathbf{k}) = tr(\tilde{N}) = \sum_{i=1}^{r} \frac{k_i}{(1-\theta)+(\theta-\lambda)k_i}.$$

Proof. Since $(X, Z) \in \mathcal{O}_n$, it suffices to derive the inverse matrix of $U(\mathbf{k})$ in (2.9), which is easily obtained by using the following well-known matrix formula for a nonsingular matrix A and a vector x,

$$(A + xx')^{-1} = A^{-1} - \frac{1}{(1 + x'A^{-1}x)} A^{-1}xx'A^{-1}.$$

(See, for example, [9, p. 592]). $\square$

The matrix $R(\mathbf{k})^{-1}$ is an M-matrix. Here, in general, an $n \times n$ matrix A is called an M-matrix if it is of the form $A = aI_n - C$, where C has nonnegative elements and $a > 0$ exceeds the absolute value of every latent root of C (see [12, p. 78]). By using the matrix identity $XX' + ZZ' = I_n$, rewrite $R(\mathbf{k})^{-1}$ as

$$R(\mathbf{k})^{-1} = \frac{1}{1-\theta} I_n - X\left\{\frac{\theta-\lambda}{1-\theta}\tilde{N} + \frac{\lambda}{1+\lambda G(\mathbf{k})}\tilde{s}\tilde{s}'\right\}X' = \frac{1}{1-\theta} I_n - C \quad \text{(say)}.$$

Since the elements of C are nonnegative and $R(\mathbf{k})$ is positive definite, $R(\mathbf{k})^{-1}$ is an M-matrix. More specifically, the (i,j)th block R^{ij} of $R(\mathbf{k})^{-1}$ is given by

$$R^{ii} = \frac{1}{1-\theta} I_{k_i} - \left\{\frac{\theta-\lambda}{(1-\theta)p(k_i)} + \frac{\lambda}{[1+\lambda G(\mathbf{k})]p(k_i)^2}\right\} 1_{k_i} 1_{k_i}',$$

$$R^{ij} = -\frac{\lambda}{[1+\lambda G(\mathbf{k})]p(k_i)p(k_j)} 1_{k_i} 1_{k_j} \quad (i \neq j),$$

where $p(k_i) = (1-\theta) + (\theta-\lambda)k_i$.

As is well known (see Theorem 4.3.2$'$ of Tong [12]), when the normality condition (3.1) holds, the matrix $R(\mathbf{k})^{-1}$ is an M-matrix if and only if the density function of Y is MTP_2 (multivariate totally positive of order 2). Hence we obtain

Theorem 3.2. *For each $\mathbf{k} \in P(n)$, the matrix $R(\mathbf{k})^{-1}$ is an M-matrix. Hence the density function of Y in* (3.1) *is MTP_2.*

The following result is a common property shared by the normal distributions whose covariance matrices are the inverse of M-matrices (see [6] or [12, Section 4.3.3]).

Corollary 3.3. *All simple, multiple and partial correlation coefficients and all linear regression coefficients calculated from Y in* (3.1) *are nonnegative.*

Next we consider the problem of comparing positive dependence.

Lemma 3.4 (A special case of [1], Theorem 5.1.6 of [12]). *Let*

$$Y = (Y_1, \ldots, Y_n)' \sim N_n(0, \Sigma) \quad \textit{and} \quad Y^* = (Y_1^*, \ldots, Y_n^*)' \sim N_n(0, \Sigma^*)$$

be such that $\Sigma = (\sigma_{ij}), \Sigma^* = (\sigma_{ij}^*) \in \mathscr{S}(n)$ and $\sigma_{ii} = \sigma_{ii}^*$ $(i = 1, \ldots, n)$. If $\sigma_{ij} \geqslant \sigma_{ij}^*$ for any $i \neq j$, and if both Σ^{-1} and Σ^{*-1} are M-matrices, then Y is more positively upper and lower orthant dependent than Y^* in absolute value, that is, both

$$P\left(\bigcap_{i=1}^n \{|Y_i| > a_i\}\right) \geqslant P\left(\bigcap_{i=1}^n \{|Y_i^*| > a_i\}\right)$$

and

$$P\left(\bigcap_{i=1}^{n}\{|Y_i|\leqslant a_i\}\right)\geqslant P\left(\bigcap_{i=1}^{n}\{|Y_i^*|\leqslant a_i\}\right)$$

hold for any $a=(a_1,\dots,a_n)'\in R^n$.

For each $\mathbf{k}\in P(n)$, let

$$R(\mathbf{k}:\theta,\lambda)=(r_{ij}(\mathbf{k}:\theta,\lambda))=(r_{ij}(\mathbf{k})).$$

Then clearly $r_{ij}(\mathbf{k}:\theta,\lambda)$ $(i\neq j)$ is increasing in θ (λ) for fixed λ (θ). Therefore by using Theorem 3.2 and Lemma 3.4, we obtain the following monotonicity result.

Theorem 3.5. *For given* $a=(a_1,\dots,a_n)'$, *the following two functions*:

$$p_U(\theta,\lambda)=P\left(\bigcap_{i=1}^{n}\{|Y_i-\beta|>a_i\}\right)\quad\textit{and}\quad p_L(\theta,\lambda)=P\left(\bigcap_{i=1}^{n}\{|Y_i-\beta|\leqslant a_i\}\right)$$

are increasing in θ (λ) *for fixed* λ (θ).

On the other hand, if the two partitions $\mathbf{k}=(k_1,\dots,k_r,0,\dots,0)'$, $\mathbf{m}=(m_1,\dots,m_p,0,\dots,0)'\in P(n)$ satisfy the following relation:

$$\begin{aligned}&\mathbf{k}\succ\mathbf{m},\\&k_1=m_1+\cdots+m_{p_1},\\&k_2=m_{p_1+1}+\cdots+m_{p_2},\\&\quad\vdots\\&k_r=m_{p_{q-1}+1}+\cdots+m_{p_q}\end{aligned}\tag{3.3}$$

for some integers $p_1,p_2,\dots,p_q$ such that $p_1<p_2<\cdots<p_{q-1}<p_q=p$, (in other words, $\mathbf{m}$ is a "finer" partition than $\mathbf{k}$), then the two matrices $R(\mathbf{k})$ and $R(\mathbf{m})$ satisfy

$$r_{ij}(\mathbf{k})\geqslant r_{ij}(\mathbf{m})\quad(i\neq j).$$

This yields

Theorem 3.6. *Let*

$$f_U(\mathbf{k})=P\left(\bigcap_{i=1}^{n}\{|Y_i-\beta|>a_i\}\right)\quad\textit{and}\quad f_L(\mathbf{k})=P\left(\bigcap_{i=1}^{n}\{|Y_i-\beta|\leqslant a_i\}\right),$$

and suppose that $\mathbf{k}$ *and* $\mathbf{m}$ *satisfies condition* (3.3). *Then the following two inequalities hold*:

$$f_U(\mathbf{k})\geqslant f_U(\mathbf{m})\quad\textit{and}\quad f_L(\mathbf{k})\geqslant f_L(\mathbf{m}).$$

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